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Nuclear electrodisintegration with nucleon emission

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Abstract. The emission of single nucleons by the scattering of electrons from nuclei is investigated. Centre-of-mass and relative coordinates are used since this avoids the Gartenhaus–Schwartz transformation and associated difficulties. The Hamiltonian describing the interaction between the electron and the nucleons is expanded in multipoles, and it is found possible to perform several of the summations over final-state angular momentum quantum numbers. The resulting formulae are applied to the electrodisintegration of ^{12}C, and comparison is made with experiment. It is of interest that, as regards the position of the maximum in the cross section, agreement with experiment is improved by introduction of a final-state interaction, which is taken to be real and central. However, the magnitude of the cross section is about twice as large as the experimental value, even allowing for uncertainties in the experimental result.

1. Introduction

The theory of the excitation of nuclei by electron scattering has been the subject of many investigations and of many reviews, for example that of de Forest and Walecka (1966), which also gives numerous references. Here we shall consider electrodisintegration, in particular with the emission of a single nucleon. A model is used which obeys antisymmetrization requirements and in which the motion of the centre of mass is included consistently. The Gartenhaus–Schwartz (1957) transformation facilitates the formulation of translationally invariant wave functions, but the compensating penalty is that it replaces one-particle operators by many-particle operators (Murray and Strachan 1966). In addition, the transformation is awkward for states which are not bound. Here we choose a model which makes the Gartenhaus–Schwartz transformation unnecessary.

The interaction between the electron and nucleus is expanded in multipoles. Numerical results are derived for ^{12}C and comparison is made with experiment. The effect of final-state interactions between the emitted nucleon and the residual nucleus is considered. Resonances in the cross section occur at energy transfers just greater than the thresholds for disintegration.

2. Kinematics

The process of interest is shown in figure 1, and this figure will serve to define the momenta and energies occurring in the collision. The rest mass of the electron is assumed to be negligible in comparison with the kinetic energies of the electron and we shall use units such that $\hbar = 1$, $c = 1$, 10^{-13} cm = 1 fm = 1. We take D to be the energy required to break N into fragments N_1 and N_2 in their lowest energy states and E_x to be the excitation

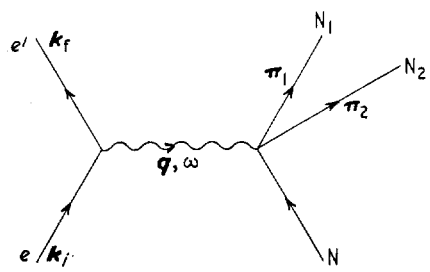

Figure 1.

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energy of the fragments. We shall use centre-of-mass $\boldsymbol{R}_A$, and relative coordinates $\boldsymbol{r}$, thereby avoiding the inconvenience of a direction-dependent energy for N_1 and also the use of the Gartenhaus–Schwartz transformation. In terms of μ, the reduced mass, and $\boldsymbol{p}$, the reduced momentum of the system,

$$\omega = D + E_x + \frac{q^2}{2M_A} + \frac{p^2}{2\mu} \tag{1}$$

where M_A is the mass of the original nucleus of A nucleons. As the commutation relations of the various momentum and coordinate variables are standard we shall not quote them. The total Hamiltonian is then

$$H = \frac{P_A^2}{2M_A} + H_c + H_0 + H_{int} \tag{2}$$

where $\boldsymbol{P}_A$ is the total momentum of the A nucleons, $H_c = T_c + V_c$ is the internal Hamiltonian for the $A-1$ nucleons of the residual nucleus,

$$H_0 = \frac{p^2}{2\mu} + V_0(r) \tag{3}$$

the Hamiltonian describing the motion of the single nucleon relative to the centre of mass of the residual nucleus. The notation implies that V_0 is a central interaction: non-central and residual interactions may be included in H_{int}. The eigenkets of $H - H_{int}$ are

$$|\boldsymbol{P}_A' cn\rangle = |\boldsymbol{P}_A'\rangle|c\rangle|n\rangle \tag{4}$$

where $\boldsymbol{P}_A'$, c, n label the eigenkets of $\boldsymbol{P}_A$, H_c, H_0. Since H_{int} depends only on the internal properties of the nuclear system an eigenket of H may be written, using G as a disposable index,

$$|\boldsymbol{P}_A' G\rangle = \sum_{c,n} A(cn; G)|\boldsymbol{P}_A' cn\rangle. \tag{5}$$

In evaluating the transition probability we sum the squares of the moduli of matrix elements of the electronic interaction over all final states $|\boldsymbol{P}_A' cn\rangle$ of the residual nucleus. We shall assume that H_{int} may be neglected in the final states.

The states of the residual nucleus will now be represented by kets $|cJ_cM_c\rangle$, where M_c is the z component of the total angular momentum J_c of the residual nucleus, and c is a set of further quantum numbers required to specify the states. These kets will be used to represent the states of the residual nucleus and also to represent the states of these same $A-1$ nucleons before ejection of the single particle. The states of the single particle bound to the $A-1$ nucleons will be represented by $|nljm\rangle$, where m is the z component of the angular momentum j and $l = j \pm \frac{1}{2}$ is the orbital angular momentum. Any additional quantum numbers which may be required are denoted by n. The unbound single-particle states will be represented similarly by $|Eljm\rangle$.

The unbound single-particle states $|Eljm\rangle$ satisfy

$$H_0|Eljm\rangle = E|Eljm\rangle. \tag{6}$$

The radial part of the corresponding wave function is

$$R_{El}(r) = aj_l(pr) + bn_l(pr), \quad p = (2\mu E)^{1/2} \tag{7}$$

for $r > r_0$, a region in which $V_0(r)$ is assumed to be zero. We have

$$\int_0^\infty R_{E'l}^*(r) R_{El}(r) r^2 \, dr = (|a|^2 + |b|^2) \frac{\pi}{2p^2} \delta(p - p'). \tag{8}$$

We set $|a|^2 + |b|^2 = 1$ in all cases.

In writing kets of the nuclear system we shall leave out the eigenkets of total linear momentum since they always give a δ-function in momentum. The basic kets for the final nuclear system are required to belong to definite total angular momentum, and we write

$$|cJ_c; Elj; J_F M_F\rangle = \sum_{M_c m} (J_c M_c jm | J_c j J_F M_F) |cJ_c M_c\rangle |Eljm\rangle. \tag{9}$$

Similarly basic kets for the initial state are

$$|cJ_c; nlj; J_I M_I\rangle = \sum_{M_c m} (J_c M_c jm | J_c j J_I M_I) |cJ_c M_c\rangle |nljm\rangle \tag{10}$$

and the complete initial state will be

$$|I; J_I M_I\rangle = \sum_{cJ_c nlj} A(cJ_c; nlj; I) |cJ_c; nlj; J_I M_I\rangle \tag{11}$$

where the coefficients $A(cJ_c; nlj; I)$ are obtained by diagonalizing the matrix for H in the space spanned by $|cJ_c; nlj; J_I M_I\rangle$. It is to be expected that in practice only a few such basic kets will be considered, so that H is diagonalized in a very limited subspace as in the Nilsson (1955) model or in the excited core model of Thankappan (1966). It can be shown (Messiah 1961, p. 734) that the density of final states is

$$\rho_N(E_f) = \frac{2p\mu}{\pi} \tag{12}$$

for the nuclear system. The density of states for the electron is well known.

3. The interaction Hamiltonian

The interaction Hamiltonian H' corresponding to the exchange of one quantum between electron and nucleus has been given by McVoy and Van Hove (1962) and we use the notation given there. The $\boldsymbol{r}_j$ and $\boldsymbol{p}_j$ are not suitable operators in our treatment and we therefore express them in terms of centre-of-mass and relative variables together with $\boldsymbol{R}_{A-1}, \boldsymbol{P}_{A-1}$ referring to the residual nucleus and $A-2$ coordinates and momenta internal to the residual nucleus.

We first consider the part of H' referring to particle 1, i.e. the ejected nucleon. Since

$$\boldsymbol{r}_1 = (\boldsymbol{r}_1 - \boldsymbol{R}_A) + \boldsymbol{R}_A = \boldsymbol{R}_A + \frac{M_{A-1}}{M_A} \boldsymbol{r} \tag{13}$$

a matrix element of $\exp(i\boldsymbol{q}.\boldsymbol{r}_1)$ between a final state $|\boldsymbol{P}_A' c' n'\rangle$ and the initial state of (11) is

$$\langle \boldsymbol{P}_A'' c' n' | \exp(i\boldsymbol{q} \cdot \boldsymbol{r}) | \boldsymbol{P}_A' : I\rangle = \delta(\boldsymbol{P}_A - '\boldsymbol{q} - \boldsymbol{P}_A') \sum_n \langle n' | \exp(i\boldsymbol{Q} \cdot \boldsymbol{r}) | n\rangle A(c'n; I). \tag{14}$$

The operator contains no part which operates on the states of the $A-1$ particles, and so the matrix element has reduced to some single-particle matrix elements. The state of the other $A-1$ particles cannot change; if $A(c'n; I) = 0$ the above matrix element is zero. We have defined

$$\boldsymbol{Q} = \boldsymbol{q} \frac{M_{A-1}}{M_A} \equiv \boldsymbol{q}\left(1 - \frac{1}{A}\right). \tag{15}$$

The remainder of the analysis is along straightforward lines and is given by Watt (1967). At present we consider only interactions with the nucleon to be expelled, i.e. direct interactions. The other interactions will be examined in a subsequent paper (Watt 1969) and will be shown to be small.

Considering only direct interactions, and omitting the centre-of-mass dependence, we have

$$H' = \left\{e+\frac{{q_\mu}^2}{8M^2}(e-2\mu)\right\}\exp(\mathrm{i}\boldsymbol{Q}\cdot\boldsymbol{r})$$
$$-\boldsymbol{\alpha}\cdot\frac{e}{2M}\left\{\boldsymbol{p}\exp(\mathrm{i}\boldsymbol{Q}\cdot\boldsymbol{r})+\exp(\mathrm{i}\boldsymbol{Q}\cdot\boldsymbol{r})\boldsymbol{p}+\frac{M}{M_\mathrm{A}}\boldsymbol{q}\exp(\mathrm{i}\boldsymbol{Q}\cdot\boldsymbol{r})\right\}.$$
$$-\boldsymbol{\alpha}\cdot\frac{\mu}{2M}\left(\frac{\boldsymbol{\sigma}\times\nabla}{1-1/A}\right)\exp(\mathrm{i}\boldsymbol{Q}\cdot\boldsymbol{r}). \tag{16}$$

We have written e, μ, $\boldsymbol{\sigma}$ for the charge, magnetic moment and spin operators for the single particle, and have assumed that $\boldsymbol{P}_\mathrm{A}' = 0$, i.e. that the initial nucleus is at rest in the laboratory. The interaction H' may now be expanded in multipoles in well-known form and the cross section obtained in terms of two form factors (de Forest and Walecka 1966).

Finally, we obtain

$$\sum_{\substack{M_\mathrm{I}M_\mathrm{F}\\ S_\mathrm{i}S_\mathrm{f}}}|\langle J_\mathrm{F}M_\mathrm{F}S_\mathrm{f}|H'|J_\mathrm{I}M_\mathrm{I}S_\mathrm{i}\rangle|^2 = 4\pi\sum_\lambda[|(J_\mathrm{F}||M(c\lambda)||J_\mathrm{I})|^2V_\mathrm{L}(\theta)$$
$$+\{|(J_\mathrm{F}||M(E\lambda)||J_\mathrm{I})|^2$$
$$+|(J_\mathrm{F}||M(M\lambda)||J_\mathrm{I})|^2\}V_\mathrm{T}(\theta)] \tag{17}$$

where $|S_\mathrm{i}\rangle$ and $|S_\mathrm{f}\rangle$ are spinors for initial and final electron states, $M(\lambda)$ are the usual multipole operators and V_L, V_T are the usual angle factors.

4. The cross section

The cross section may be written as

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}k_\mathrm{f}\,\mathrm{d}\Omega_\mathrm{f}} = \sum_{J_\mathrm{F}l_j}\frac{\mathrm{d}^2\sigma'}{\mathrm{d}k_\mathrm{f}\,\mathrm{d}\Omega'} = \sigma_M\left\{|F_\mathrm{L}|^2+\left(\frac{{q_\mu}^2}{2q^2}+\tan^2\tfrac{1}{2}\theta\right)|F_\mathrm{T}|^2\right\} \tag{18}$$

where

$$\sigma_M = \frac{4\alpha^2{k_\mathrm{f}}^2}{{q_\mu}^4}\cos^2\tfrac{1}{2}\theta = \frac{\alpha^2\cos^2\tfrac{1}{2}\theta}{4{k_\mathrm{i}}^2\sin^4\tfrac{1}{2}\theta} \tag{19}$$

is the Mott cross section for scattering from a point charge,

$$|F_\mathrm{L}|^2 = \sum_{J_\mathrm{F}jl}4\pi\sum_\lambda\frac{|(cJ_c;\,Elj;\,J_\mathrm{F}||M(c\lambda)||I;\,J_\mathrm{I})|^2f^2({q_\mu}^2)}{2J_\mathrm{I}+1}\rho_\mathrm{N}(E) \tag{20}$$

$$|F_\mathrm{T}|^2 = \sum_{J_\mathrm{F}jl\lambda}4\pi\left\{\frac{|(cJ_c;\,nlj;\,J_\mathrm{F}||M(E\lambda)||I;\,J_\mathrm{I})|^2}{2J_\mathrm{I}+1}\right.$$
$$\left.+\frac{|(cJ_c;\,Elj;\,J_\mathrm{F}||M(M\lambda)||I;\,J_\mathrm{I})|^2}{2J_\mathrm{I}+1}\right\}f^2\rho_\mathrm{N}(E). \tag{21}$$

F_L and F_T will be called the longitudinal and transverse form factors.

For any single-particle tensor operator of rank λ,

$$\{cJ_c;\,Elj;\,J_\mathrm{F}||M(\lambda)||I;J_\mathrm{I}\} = \sum_{c'J_c'n'l'j'}A(c'J_c';\,n'l'j';\,I)\,(cJ_c;\,Elj;\,J_\mathrm{F}||M(\lambda)||c'J_c';\,n'l'j';\,J_\mathrm{I})$$
$$= \sum_{c'J_c'n'l'j'}A(c'J_c';\,n'l'j';\,I)\,\delta_{cc'}\,\delta_{J_cJ_c'}\,(-)^{J_c+j'+J_\mathrm{F}+\lambda}$$
$$\times\{(2J_\mathrm{F}+1)(2J_\mathrm{I}+1)\}^{1/2}\begin{Bmatrix} j & J_\mathrm{F} & J_c\\ J_\mathrm{I} & j' & \lambda\end{Bmatrix}(Elj||M(\lambda)||n'l'j')$$

$$= \sum_{n'l'j'} A(cJ_c; n'l'j'; I)(-)^{J_c+j'+J_F+\lambda}\{(2J_F+1)(2J_I+1)\}^{1/2} \times \begin{Bmatrix} j & J_F & J_c \\ J_I & j' & \lambda \end{Bmatrix} (Elj||M(\lambda)||n'l'j') \,. \tag{22}$$

Using the result (Edmonds 1957)

$$\sum_{J_F}(2J_F+1)\begin{Bmatrix} j & J_F & J_c \\ J_I & j' & \lambda \end{Bmatrix}\begin{Bmatrix} j & J_F & J_c \\ J_I & j'' & \lambda \end{Bmatrix} = \frac{1}{2j'+1}\delta_{j'j''} \tag{23}$$

we obtain

$$\sum_{J_F jl}\frac{|(cJ_c; Elj; J_F||M(\lambda)||I; J_I)|^2}{2J_I+1} = \sum_{\substack{n'l'j'lj \\ n''l''}} A(cJ_c; n'l'j'; I)A^*(cJ_c; n''l''j'; I) \times \frac{(-)^{2(J_c+j'+J_F+\lambda)}}{2j'+1}(Elj||M(\lambda)||n'l'j') \times (Elj||M(\lambda)||n''l''j')^*. \tag{24}$$

In all cases the reduced matrix elements of the single-particle operators appearing in equation (24) may be expressed entirely in terms of radial integrals. However, the algebra is different for each type of operator, and we shall consider only the case of the Coulomb operator in detail. Here the operator acts only on the spatial coordinates and not on the spin, so that (Edmonds 1957)

$$(Elj||M(c\lambda)||n'l'j') = (-)^{l+\frac{1}{2}+j'+\lambda}\{(2j+1)(2j'+1)\}^{1/2}\begin{Bmatrix} l & j & \frac{1}{2} \\ j' & l' & \lambda \end{Bmatrix}(El||M(c\lambda)||n'l'). \tag{25}$$

Hence

$$\sum_{J_F jl}\frac{|(J_F||M(c\lambda)||J_I)|^2}{2J_I+1} = \sum_{\substack{n'l'j'lj \\ n''l''}} A(cJ_c; n'l'j'; I)A^*(cJ_c; n''l''j'; I)\frac{1}{2j'+1} \times (2j+1)(2j'+1)(-)^{2(j'+\frac{1}{2}+\lambda+l)}\begin{Bmatrix} l & j & \frac{1}{2} \\ j' & l' & \lambda \end{Bmatrix}\begin{Bmatrix} l & j & \frac{1}{2} \\ j' & l'' & \lambda \end{Bmatrix} \times (El||M(c\lambda)||n'l')(El||M(c\lambda)||n''l''). \tag{26}$$

The summation over j is now possible:

$$\sum_{J_F jl}\frac{|(J_F||M(c\lambda)||J_I)|^2}{2J_I+1} = \sum_{\substack{n'l'j' \\ n''l}} A(cJ_c; n'l'j'; I)\,A^*(cJ_c; n''l'j'; I) \times \frac{1}{2l'+1}(El||M(c\lambda)||n'l')\,(El||M(c\lambda)||n''l')^*. \tag{27}$$

In this last expression the reduced matrix elements are integrals over the spatial coordinates. Using the explicit form for the Coulomb operator and the expression for the reduced matrix element of a spherical harmonic (Edmonds 1957),

$$(El||M(c\lambda)||n'l') = \int R^*_{El}(r)\left[\frac{q_\mu{}^2}{q^2}\left\{e+\frac{q^2}{8M^2}(e-2\mu)\right\}\mathrm{j}_\lambda(Qr)\right]R_{n'l'}(r)r^2\,\mathrm{d}r \times (-)^l\left\{\frac{(2l+1)(2l'+1)(2\lambda+1)}{4\pi}\right\}^{1/2}\begin{pmatrix} l & \lambda & l' \\ 0 & 0 & 0 \end{pmatrix}. \tag{28}$$

Hence

$$\sum_{J_F jl}\frac{|(J_F||M(c\lambda)||J_I)|^2}{2J_I+1} = \sum_{\substack{n'l'j' \\ n''l}} A(cJ_c; n'l'j'; I)\,A^*(cJ_c; n''l'j'; I) \times (El|\mathscr{J}(c\lambda)|n'l')(El|\mathscr{J}(c\lambda)|n''l')\,X^{(c)}(l\lambda l'). \tag{29}$$

We have introduced the notation

$$X^{(c)}(l\lambda l') = \frac{1}{2l'+1}\left\{\frac{(2l+1)(2\lambda+1)(2l'+1)}{4\pi}\right\}\begin{pmatrix} l & \lambda & l' \\ 0 & 0 & 0 \end{pmatrix}^2 \tag{30}$$

and

$$\mathscr{J}(c\lambda) = \frac{q_\mu^2}{q^2}\left\{e+\frac{q^2}{8M^2}(e-2\mu)\right\} j_\lambda(Qr). \tag{31}$$

The coefficient $X^{(c)}$ is completely determined, and has appeared from the angular momentum algebra and the reduced matrix elements of the spin and angular functions.

We shall assume that, for a given value of l', different values of n' do not occur. Hence

$$\sum_{J_F j l} |(cJ_c; Elj; J_F||M(c\lambda)||I; J_I)|^2 = \sum_{\substack{n'l'j' \\ l}} |A(cJ_c; n'l'j'; I)|^2 |(El|\mathscr{J}(c\lambda)|n'l')|^2 X^{(c)}(l\lambda l'). \tag{32}$$

The summation over l is strictly limited by the angular momentum coefficient which contains $\begin{pmatrix} l & \lambda & l' \\ 0 & 0 & 0 \end{pmatrix}$. Thus $l = \lambda+l',\ \lambda+l'-1,\ \ldots,\ \lambda-l'$ by the usual triangle inequality, but also $l+\lambda+l'$ must be even, for $\begin{pmatrix} l & \lambda & l' \\ 0 & 0 & 0 \end{pmatrix}$ vanishes otherwise. For $l' = 0$, only $l = \lambda$ contributes, while for $l' = 1$, only $l = \lambda \pm 1$ give non-zero terms. For given n' and l', $j' = l' \pm \frac{1}{2}$ and, even in quite complicated nuclear models, only one or two sets $\{n'\ l'\ j'\}$ will appear. In practice, therefore, the summations in equation (32) are expected to present little trouble.

To evaluate F_L we must sum over λ, which goes from 0 to ∞. It turns out, in the calculations performed here, that the terms fall off very rapidly, and the first five terms contain most of the sum.

The electric and magnetic reduced matrix elements may be written in a form analogous to expression (32). The calculation of the $X^{(e)}$ and $X^{(m)}$ coefficients is more complicated than the calculation of $X^{(c)}$. However, the summation over j may be performed in all cases, giving a $\delta_{l'l''}$ exactly as for $X^{(c)}$. Also it has been assumed that, for given l', only one n' occurs, just as before. This assumption is, of course, by no means necessary. It is used merely to simplify the form of equation (30). The angular momentum factors which we require are unaffected by this assumption.

Because of the complicated form of the $X^{(e)}$ and $X^{(m)}$ coefficients, they will not be given here. They are given in a thesis by one of the authors (Watt 1967). In the particular case of ^{12}C, which is assumed to have two protons and two neutrons in the $1s_{1/2}$ shell and four protons and four neutrons in the $1p_{3/2}$ shell, the total cross section will be

$$\frac{d^2\sigma}{dk_f\, d\Omega_f} = 2\sigma_{1s_{1/2}}(p)+2\sigma_{1s_{1/2}}(n)+4\sigma_{1p_{3/2}}(p)+4\sigma_{1p_{3/2}}(n). \tag{33}$$

The results of the simple model are correct for shells with one particle only and for closed shells. In other cases antisymmetrization is important.

5. No final-state interaction

It is convenient to collect the results which are obtained if one assumes that the emitted nucleon does not interact with the residual nucleus. The nucleon wave function may be taken as a plane wave, $\exp(i\boldsymbol{p}.\boldsymbol{r})$ in the final state, and we shall use at present the harmonic oscillator 1s-shell wave function $(\kappa/\pi)^{3/4}\exp(-\frac{1}{2}\kappa r^2)$ for the initial state. When matrix elements of H' (equation (16)) are taken between these initial and final states, and, if we sum over final and average over initial electron states, the cross section may be written

$$\frac{d^3\sigma'}{dk_f\, d\Omega_f\, d\Omega_p} = \sigma_M\left\{|F_L'|^2+\left(\frac{q_\mu^2}{2q^2}+\tan^2\tfrac{1}{2}\theta\right)|F_T'|^2\right\} \tag{34}$$

where σ_M is the Mott scattering cross section and F_L' and F_T' will be referred to as longitudinal and transverse form factors respectively. The density of final states for the nuclear system is now

$$\rho_N = \frac{\mu}{p}\frac{d^3\boldsymbol{p}}{(2\pi)^3}. \tag{35}$$

Since

$$|F_L'|^2 \propto \left|\int \exp(-i\boldsymbol{p}.\boldsymbol{r}) \exp(i\boldsymbol{Q}.\boldsymbol{r}) \exp(-\tfrac{1}{2}\kappa r^2)\, d^3r\right|^2$$
$$\propto \exp\left\{-\frac{(\boldsymbol{p}-\boldsymbol{Q})^2}{\kappa}\right\} \tag{36}$$

we obtain, after integrating over directions of $\boldsymbol{p}$,

$$\frac{d^2\sigma}{dk_f\, d\Omega_f} = \sigma_M \left\{|F_L|^2 + \left(\frac{q_\mu^2}{2q^2} + \tan^2\tfrac{1}{2}\theta\right)|F_T|^2\right\} \tag{37}$$

where

$$|F_L|^2 = \int |F_L'|^2\, d\Omega_p \propto \exp\left(-\frac{p^2+Q^2}{\kappa}\right)\frac{2\pi\kappa}{pQ}\sinh\left(\frac{2pQ}{\kappa}\right). \tag{38}$$

Now, in our problem, $2pQ/\kappa \gg 1$, so that we replace $\sinh(2pQ/\kappa)$ by $\frac{1}{2}\exp(2pQ/\kappa)$, obtaining

$$|F_L|^2 = \frac{2\mu}{Q(\kappa\pi)^{1/2}}\exp\left\{-\frac{(p-Q)^2}{\kappa}\right\} f^2(q^2)\left[\frac{q_\mu^2}{q^2}\left\{e_1 + \frac{q^2}{8M^2}(e_1 - 2\mu_1)\right\}\right]^2 \tag{39}$$

for the square of the longitudinal form factor for two identical particles of charge e_1 and magnetic moment μ_1 filling the 1s shell. If we make the same approximation as above, the transverse form factor is

$$|F_T|^2 = \frac{4\mu}{Q(\kappa\pi)^{1/2}} f^2(q^2)\frac{1}{M^2}\left\{e_1\frac{p\kappa}{2Q}\left(1 - \frac{\kappa}{2pQ}\right) + \frac{\mu_1^2 q^2}{4}\right\}\exp\left\{-\frac{(p-Q)^2}{\kappa}\right\}. \tag{40}$$

In our calculations on ^{12}C, $p \lesssim 2$, $\kappa \simeq 0{\cdot}3$, $Q \simeq 1{\cdot}2$. Hence the first term in curly brackets is much smaller than the other. This agrees with Czyz's conclusion that the current part of the operator is much less important than the magnetic moment part (Czyz 1963).

Thus, to a good approximation ($\sim 10\%$), the cross section for 1s-shell particles is

$$\frac{d^2\sigma}{dk_f\, d\Omega_f} = \exp\left\{-\frac{(p-Q)^2}{\kappa}\right\}\Sigma_s \tag{41}$$

where Σ_s is independent of p. Similarly, for 1p-shell nucleons

$$\frac{d^2\sigma}{dk_f\, d\Omega_f} = \left\{1 + \frac{(p-Q)^2}{\kappa}\right\}\exp\left\{-\frac{(p-Q)^2}{\kappa}\right\}\Sigma_p\,. \tag{42}$$

In the circumstances considered later, Q and hence Σ_s, Σ_p are reasonably constant over the entire spectrum, while p varies from 0 to about 2, and so the shape of the spectrum is determined largely by the $p-Q$ terms in the above expressions.

6. Electrodisintegration of carbon

The quasi-elastic peak has been studied in carbon, and so we have performed calculations for this nucleus. We choose the simplest shell model to describe the nucleus, assuming that the $1s_{1/2}$ and $1p_{3/2}$ shells are completely filled and that all other shells are empty. We are therefore interested in coefficients of the type appearing in equation (30) with $l' = 0, 1,$

and these may be simplified by evaluation of the 3–j and 6–j symbols. The results are given below.

$$l' = 0$$

$$X^{(c)} = \frac{2\lambda+1}{4\pi}\delta_{l,\lambda} \qquad X_1^{(m)} = \frac{1}{4\pi}\frac{\lambda}{4}\mu^2\delta_{l,\lambda+1}$$

$$X_1^{(e)} = \frac{2\lambda+1}{\lambda(\lambda+1)}\frac{1}{4\pi}\delta_{l,\lambda} \qquad X_2^{(m)} = \frac{1}{4\pi}\frac{\lambda+1}{4}\mu^2\delta_{l\ \lambda-1}$$

$$X_2^{(e)} = \frac{2\lambda+1}{4\pi}\delta_{l,\lambda} \qquad X_3^{(m)} = X_3^{(e)} = 0.$$

$$l' = 1$$

$$X^{(c)} = \frac{\lambda+1}{4\pi}\delta_{l,\lambda+1} + \frac{\lambda}{4\pi}\delta_{l,\lambda-1}$$

$$X_1^{(e)} = \frac{1}{4\pi}\frac{1}{\lambda}\delta_{l,\lambda+1} + \frac{1}{4\pi}\frac{1}{\lambda+1}\delta_{l,\lambda-1}$$

$$X_2^{(e)} = \frac{1}{4\pi}\left\{\lambda+1+\frac{2}{2j+1}(-)^{j+\frac{1}{2}}\right\}\delta_{l,\lambda+1} + \frac{1}{4\pi}\left\{\lambda+\frac{2}{2j+1}(-)^{j-\frac{1}{2}}\right\}\delta_{l,\lambda-1}$$

$$X_3^{(e)} = \frac{1}{4\pi}\left\{j(j+1)-\frac{11}{4}\right\}(\delta_{l,\lambda+1}-\delta_{l,\lambda-1})$$

$$X_1^{(m)} = \frac{1}{4\pi}\mu^2\frac{3\lambda(\lambda+2)}{8(2\lambda+3)}\delta_{l,\lambda+2}\delta_{j,\frac{3}{2}}$$
$$+\frac{1}{4\pi}\left[e^2\left(1-\frac{1}{A}\right)^2\frac{1}{2\lambda+1}+\mu^2\frac{\lambda(\lambda+1)}{4(2\lambda+3)}\left\{1+\frac{(-)^{j-\frac{1}{2}}2(\lambda+2)}{(2j+1)(\lambda+1)}\right\}\right.$$
$$\left.+e\mu\left(1-\frac{1}{A}\right)\frac{\lambda}{2(2\lambda+1)}\left\{\frac{11}{4}-j(j+1)\right\}\right]\delta_{l,\lambda}$$

$$X_2^{(m)} = \frac{1}{4\pi}\left(e^2\left(1-\frac{1}{A}\right)^2\frac{1}{2\lambda+1}+\mu^2\frac{3}{8}\frac{\lambda+1}{2j+1}\left[1+\frac{(-)^{j+\frac{1}{2}}\{j(j+1)-\lambda-\frac{1}{4}\}}{3(\lambda-\frac{1}{2})}\right]\right.$$
$$\left.-e\mu\left(1-\frac{1}{A}\right)\frac{\lambda+1}{2(2\lambda+1)}\left\{\frac{11}{4}-j(j+1)\right\}\right)\delta_{l,\lambda}$$
$$+\frac{1}{4\pi}\mu^2\frac{3}{8}\frac{(\lambda-1)(\lambda+1)}{2\lambda-1}\delta_{l,\lambda-2}\delta_{j,\frac{3}{2}}$$

$$X_3^{(m)} = \frac{1}{4\pi}\frac{2}{2\lambda+1}\left[e^2\left(1-\frac{1}{A}\right)^2-e\mu\left(1-\frac{1}{A}\right)\frac{1}{4}\left\{\frac{11}{4}-j(j+1)\right\}\right]\delta_{l,\lambda}. \tag{43}$$

The radial integrals such as expression (28) may be evaluated as soon as initial- and final-state wave functions are found. Harmonic oscillator wave functions were used for the initial state. Final-state wave functions were found by assuming that the nucleon–nucleus interaction could be described by a finite real central potential well. A solution of the resulting differential equation was found numerically.

The required integrals may be expressed in terms of the set

$$I(l,\lambda,c) = \int_0^\infty R_{El}(r)\mathrm{j}_\lambda(Qr)\exp(-\tfrac{1}{2}\kappa r^2)r^c\,\mathrm{d}r \tag{44}$$

where $R_{El}(r)$ is the radial part of the wave function for the emitted particle, and κ is the harmonic oscillator constant. For example, $n' = 1$ and $l' = 0$ for 1s-shell nucleons, and so

$$(El|(\mathscr{J}c\lambda)|10) = 2\sqrt{\kappa}\left(\frac{\kappa}{\pi}\right)^{1/4}\frac{q_\mu{}^2}{q^2}\left\{e+\frac{q^2}{8M^2}(e-2\mu)\right\}I(l\lambda 2). \tag{45}$$

Using

$$\mathrm{j}_{\lambda+1}(\kappa)+\mathrm{j}_{\lambda-1}(\kappa) = \frac{2\lambda+1}{\kappa}\mathrm{j}_\lambda(\kappa) \tag{46}$$

we obtain, for example,

$$I(l,\lambda,2) = \frac{Q}{2\lambda+1}\{I(l,\lambda-1,3)+I(l,\lambda+1,3)\}. \tag{47}$$

By using this and similar relations, all the required integrals may be found from the sets $I(\lambda,\ \lambda\pm1,\ 3)$ and $I(\lambda,\ \lambda\pm1,\ 5)$, $\lambda = 0, 1, \ldots \infty$. These integrals were evaluated numerically.

The harmonic oscillator constant κ is found to be 0·364 from elastic electron scattering (Ehrenberg 1959, Crannell 1966). In the usual shell model each nucleon moves in a potential well fixed in space, the centre of the well being fixed at the mean position of the centre of mass, $\boldsymbol{R}_\mathrm{A}$. However, we are interested in relative coordinates, $\boldsymbol{r} = \boldsymbol{r}_1-\boldsymbol{R}_{\mathrm{A}-1}$, i.e. $\boldsymbol{r} = \{A/(A-1)\}(\boldsymbol{r}-\boldsymbol{R}_\mathrm{A})$, and so we take

$$\kappa = \left(\frac{A-1}{A}\right)^2 0{\cdot}364 = 0{\cdot}3. \tag{48}$$

This adjustment is similar to that given by Tassie and Barker (1958) for the harmonic oscillator model.

The effect of varying κ has not been investigated in great detail, but calculations for $\kappa = 0{\cdot}4$ give a hump which is slightly flatter and broader than that for $\kappa = 0{\cdot}3$. These conclusions agree with those of Griffy *et al.* (1966).

The binding energies were taken to be 15·9 MeV for 1p-shell protons, 18·7 MeV for 1p-shell neutrons, 30·9 MeV for 1s-shell protons and 33·7 MeV for 1s-shell neutrons (Ajzenberg and Lauritsen 1955, Jacob and Maris 1966). The nucleon form factors were taken to be (Elton 1961)

$$f(q^2) = \exp\left(-\frac{q^2}{4}0{\cdot}43\right). \tag{49}$$

The case of no final-state interaction discussed above was used in two ways. From equations (32) and (30) we obtain for 1s-shell particles

$$F_\mathrm{L}{}^2(\mathrm{s}) = f^2\left[\frac{q_\mu{}^2}{q^2}\left\{e+\frac{q^2}{8M^2}(e-2\mu)\right\}\right]^2\frac{2\mu p}{\pi}\left\{4\kappa\left(\frac{\kappa}{\pi}\right)^{1/2}\right\}2$$
$$\times 4\pi\sum_{\lambda=0}^{\infty}\frac{2\lambda+1}{4\pi}I^2(\lambda,\lambda,2). \tag{50}$$

Comparison with equation (39) allows us to decide how many terms of the infinite series are needed to give a good approximation to the sum. This also tells us how many terms are important in calculations of the type carried out by Eisenberg (1963). The number required varies quite rapidly with p; in our calculations, the first ten terms were found to be adequate even at large values of p ($\sim 2{\cdot}2$).

Bounin and Bishop (1961) have performed an experiment in which electrons of energy 194 MeV were scattered from carbon. The scattering angle was 135°, and only the scattered electrons were observed. Their results are illustrated in figure 2. Our arbitrary unit of cross section is expected to be 1 nbn sr^{-1} MeV^{-1}, to within about 20%.

We have considered electron scattering under these conditions to effect some comparison with experiment. Several final-state interactions were considered. The case $V = 0$ was used to allow the comparisons discussed above to be made, and to check that the programme

was working properly. In addition, square wells and Woods–Saxon potentials of different depths and radii were used.

Results for a square well of depth 20 MeV and radius 2·9 fm are given in detail in figure 3. It is interesting to note that neutron emission is quite important, contributing about 20% of the total cross section. The longitudinal form factors for neutron emission are almost (but not quite) zero, and the transverse form factors for neutrons are typically half as large as the transverse form factors for protons. The total cross section is a roughly symmetrical hump of maximum height 6·4 nbn sr^{-1} MeV^{-1} at 128 MeV, and width 44 MeV at half-maximum. There are also two sharp quite large peaks at 170 and 157 MeV respectively. These are due to single-particle resonances in the potential well. Unfortunately, direct knockout is not a good approximation at these low energies, because there will be a large probability that the nucleon will lose energy to the residual nucleus, and a more detailed approach would be required to treat this region properly (Raphael and Überall 1966, de Forest 1967). These resonances are interesting features of the cross section, but we do not concern ourselves with them here.

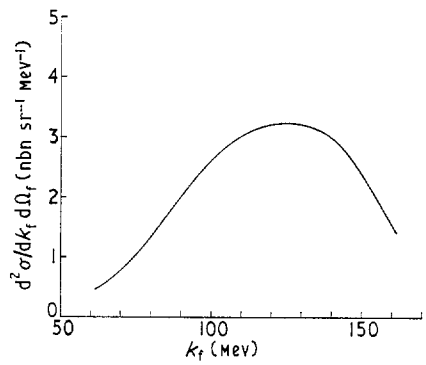

Figure 2. Experimental results for ^{12}C (Bounin and Bishop 1961). A smooth curve has been drawn by hand through their results to obtain the above graph: The abscissa gives the energy of the scattered electron, and the ordinate is calibrated in arbitrary units, which are expected to be close to 1 nbn sr^{-1} MeV^{-1}.

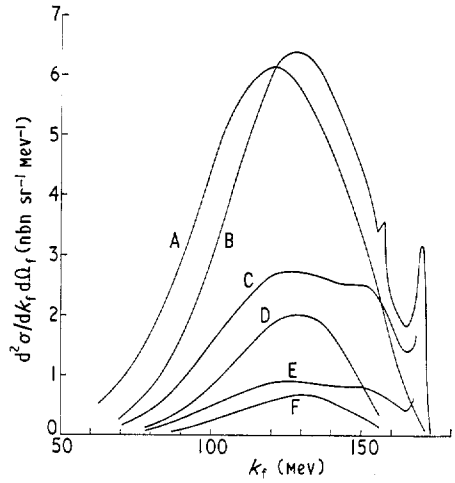

Figure 3. Calculated cross sections. Curve A gives the total cross section in the case of no final-state interaction. All other curves were obtained using a square-well interaction of radius 2·9 fm and depth 20 MeV: B, total cross section; C, D, contributions from 1p- and 1s-shell protons; E, F, contributions from 1p- and 1s-shell neutrons.

The absolute magnitude of the cross section as calculated is larger than the experimental result by a factor of about 1·8. We have also performed calculations to compare with the results of Leiss and Taylor (quoted by Czyz 1963), and the same discrepancy appears. The reason for this may be that the approximation of neglecting the two-body operators in the interaction Hamiltonian is not a good one, and that their inclusion will reduce the cross section. This is also suggested by the sum rule of McVoy and Van Hove (1962). If our cross section, obtained from equation (37), is integrated over final electron energies, only the diagonal terms of the sum rule are obtained. Inclusion of the two-body operators in the interaction Hamiltonian should give the off-diagonal terms, decreasing the sum-rule result and hence the cross section of interest here. Calculations are now in progress which include these extra terms.

We now consider the effects of different final-state interactions. The main results are collected in table 1. We have not presented graphs for all the final-state interactions considered, for they all look rather similar to the two curves shown in figure 3. The position

Table 1

Type of interaction	V (MeV)	R (fm)	a (fm)	Peak (MeV)	Cross section at peak	Width
	0			122	6·35	60
square well	20	2·9		129	6·55	49
square well	40	2·9		137	6·45	42
square well	20	2·7		129	6·55	48
Woods–Saxon	20	2·9	0·5	128	6·55	56
Woods–Saxon	40	2·9	0·5	136	6·60	44
Woods–Saxon	20	2·7	0·5	128	6·55	54
Experiment				130	3·50	70

V and R are the depths and radii of the potential wells and the 'skin thickness' of the Woods–Saxon potential is a. The form of the Woods–Saxon potential is

$$V(r) = \frac{-V}{1+\exp\{(r-R)/\}a}.$$

and width of the quasi-elastic peak is rather sensitive to the final-state interaction. It can be seen that the position of the peak moves to the right (i.e. to the region of smaller energy loss by the electron) as the depth of the potential is increased. This can be understood qualitatively as follows. If we refer again to § 5, the final-state wave function is taken to be $\exp(i\boldsymbol{p}.\boldsymbol{r})$, where $\boldsymbol{p}$ is the momentum of the nucleus far from the residual nucleus. However, in calculating the matrix elements, such as those in equation (39), it would be better to replace $\boldsymbol{p}$ by an effective momentum, $\boldsymbol{p}'$ say, inside the nucleus. But $\boldsymbol{p}'$ will be greater than $\boldsymbol{p}$, the relation being roughly

$$p'^2 = p^2 + 2MV \tag{51}$$

where V is some average depth of the potential well over the nuclear volume. Since the cross section will now be proportional to $\exp\{-(p'-Q)^2/\kappa\}$, which peaks at $p' = Q$, the peak will move to the right as V increases. It is found that, with $V = 10$ MeV in equation (51), the cross section obtained from expression (37) with p' replacing p is virtually identical with the total cross section for the square well of depth 20 MeV shown in figure 3, except, of course, that no resonances occur. Thus we can simulate final-state interactions in a simple way if great detail is not required. By choosing a well of suitable depth, it is possible to fit the position of the peak.

In all cases the width of the calculated curve is too small. The width can be fitted by using an energy-dependent potential, the depth increasing from 0 for high proton energies to about 50 MeV for low proton energies.

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